Section 6.2: One-to-One Functions; Inverse Functions

Objectives:

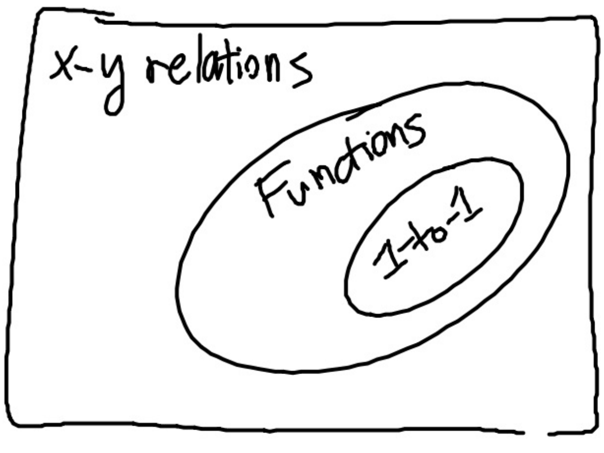
- 6.2: Determine whether a function is one-to-one.
- 6.2: Determine the inverse of a function defined by a map or a set of ordered pairs.
- 6.2: Obtain the graph of the inverse function from the graph of the function.
- 6.2: Find the inverse of a function defined by an equation.

Definition: One-to-One

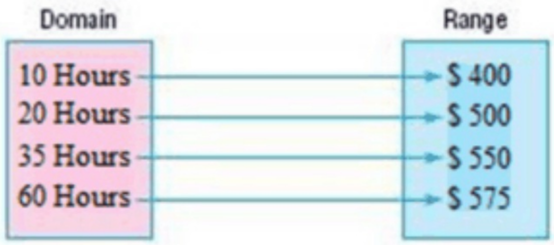
A function $f(x)$ is **one-to-one** if no two elements in the domain correspond to the same element in the range; that is,

$$\text{If } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2)$$

Each x value corresponds to exactly one y value, ***and*** each y value corresponds to exactly one x value.



Example's of One-to-One Functions



$$\{(2,6),\ (3,9),\ (-9,11),\ (6,-17)\}$$

Definition: Horizontal Line Test

If every horizontal line intersects the graph of a function in at most one point, then the function is classified as a one-to-one function.

Example 1: Determine Whether a Function is One-to-One

For each function, determine whether the function is one-to-one.

(a) $f(x) = x^2$ NO

(b) $g(x) = x^3$ Yes

other 1-to-1s?

$y = x$ (identity)

$y = \sqrt{x}$ (sq. root)

$y = \frac{1}{x}$ (reciprocal)

Definition: Inverse Function

If f and g denote two one-to-one functions such that

$$f(g(x)) = x \text{ and } g(f(x)) = x$$

then g is the **inverse** of the function f. The function g is denoted by f^{-1} (read as "f-inverse").

Ex $f(x) = x+2$
$g(x) = x-2$

Find
$f(g(5)) = 5$
$g(f(8)) = 8$

Example 2: Verifying Inverse Functions

Verify that $f^{-1}(x) = (x-1)^3 + 2$ is the inverse of $f(x) = \sqrt[3]{x-2} + 1$.

To prove they are inverses, do the composition.

$$f(g(x)) = \sqrt[3]{(x-1)^3 \cancel{+2} - \cancel{2}} + 1$$
$$= \sqrt[3]{(x-1)^3} + 1 = (x-1) + 1 = x \checkmark$$

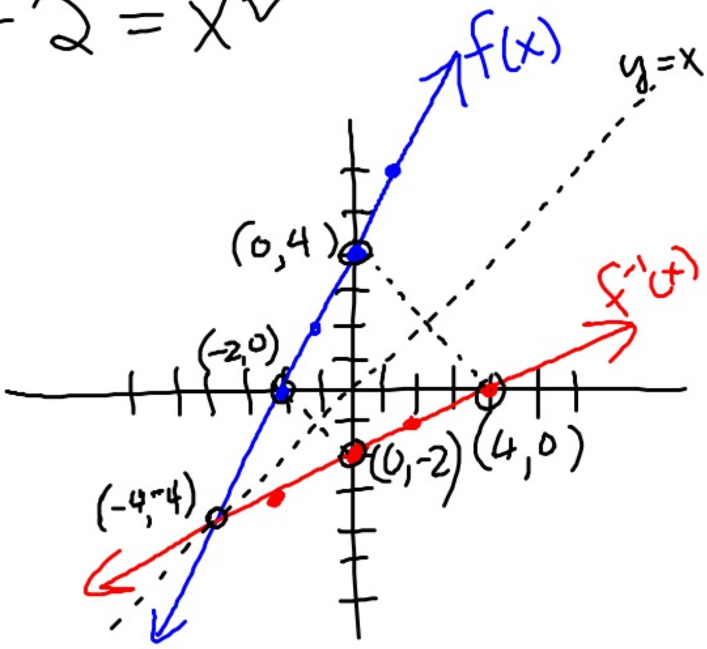
$$g(f(x)) = \left(\sqrt[3]{x-2} \cancel{+1} - \cancel{1}\right)^3 + 2$$
$$= \left(\sqrt[3]{x-2}\right)^3 + 2 = x - 2 + 2 = x \checkmark$$

Your Turn

Verify that $f^{-1}(x) = \frac{1}{2}x - 2$ is the inverse of $f(x) = 2x + 4$.

$$f(g(x)) = 2\left(\tfrac{1}{2}x - 2\right) + 4$$
$$= x - 4 + 4 = x \checkmark$$

$$g(f(x)) = \tfrac{1}{2}\left(2x + 4\right) - 2$$
$$= x + 2 - 2 = x \checkmark$$



Inverse Functions

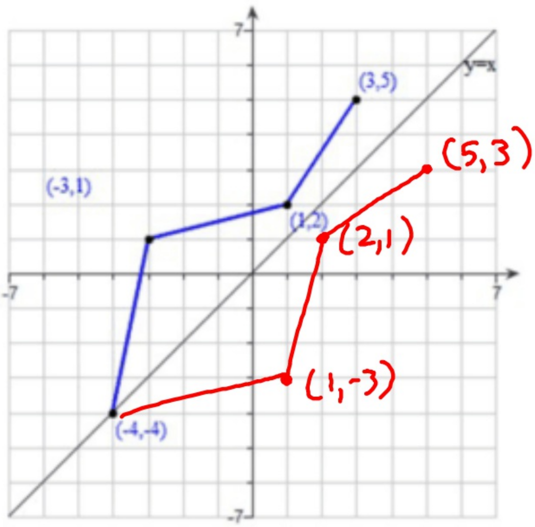
Domain of f = Range of f^{-1}
Range of f = Domain of f^{-1}

Symmetry

The graph of a function and its inverse are always symmetric about the line $y = x$.

Example 3: Obtain the Graph of the Inverse Function from the Graph of the Function

The graph of a one-to-one function f is given. Draw the graph of the inverse function f^{-1}.



Steps for Finding the Inverse Function of an Equation

- **Step 1.** Let $y = f(x)$.
- **Step 2.** Interchange x and y.
- **Step 3.** Solve for y in terms of x.
- **Step 4.** Let $y = f^{-1}(x)$.

Ex: $f(x) = 2x + 1$

$y = 2x + 1$ (step 1)

$x = 2y + 1$ (step 2)

$x - 1 = 2y$

$\frac{x-1}{2} = y$ (step 3)

$\boxed{f^{-1}(x) = \frac{x-1}{2} = \frac{1}{2}x - \frac{1}{2}}$ (step 4)

Example 4: Find the Inverse of a Function Defined by an Equation

The function $f(x) = x^3 + 7$ is one-to-one.

(a) Find the domain and range of $f(x)$. D: all reals
R: all reals

(b) Find the inverse of $f(x)$. $f^{-1}(x) = \sqrt[3]{x-7}$

(c) Find the domain and range of $f^{-1}(x)$. D: all reals
R: all reals

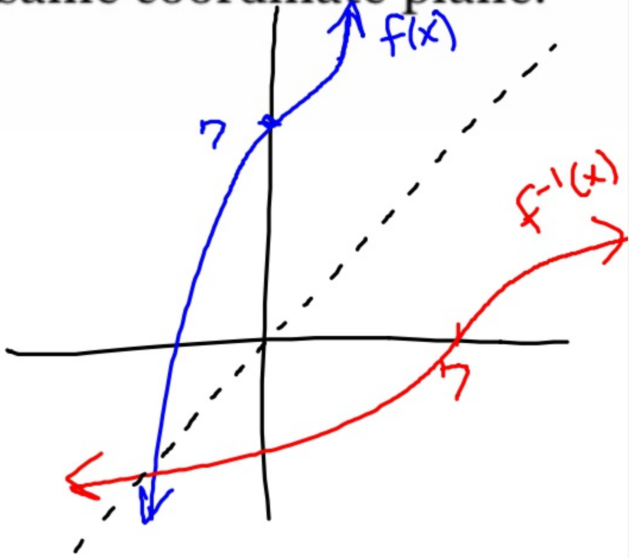
(d) Graph f and f^{-1} on the same coordinate plane.

(b) $x = y^3 + 7$

$x - 7 = y^3$

$\sqrt[3]{x-7} = y$

(d)



Example

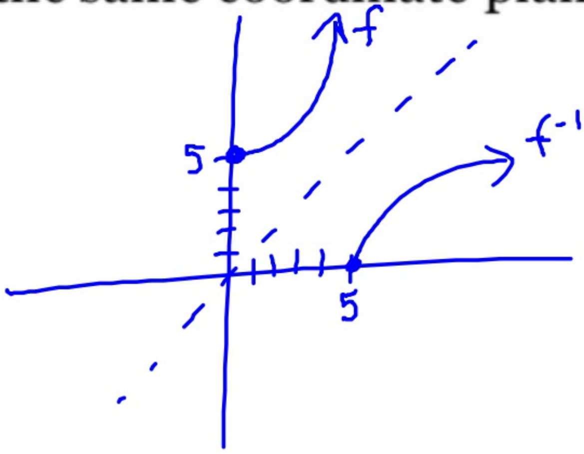
The function $f(x) = x^2 + 5,\ \underbrace{x \geq 0}_{\text{domain restricted to make it 1-to-1}}$ is one-to-one.

(a) Find the domain and range of $f(x)$. $D: x \geq 0$ $R: y \geq 5$

(b) Find the inverse of $f(x)$. $f^{-1}(x) = \sqrt{x-5}$

(c) Find the domain and range of $f^{-1}(x)$. $D: x \geq 5$ $R: y \geq 0$

(d) Graph f and f^{-1} on the same coordinate plane.



Example 5: Find the Inverse of a Function Defined by an Equation

The function $f(x) = \frac{3}{x-1}$ is one-to-one.

(a) Find the domain and range of $f(x)$. D: $x \neq 1$
R: $y \neq 0$

(b) Find the inverse of $f(x)$. $f^{-1}(x) = \frac{x+3}{x}$

(c) Find the domain and range of $f^{-1}(x)$. D: $x \neq 0$
R: $y \neq 1$

(d) Graph f and f^{-1} on the same coordinate plane.

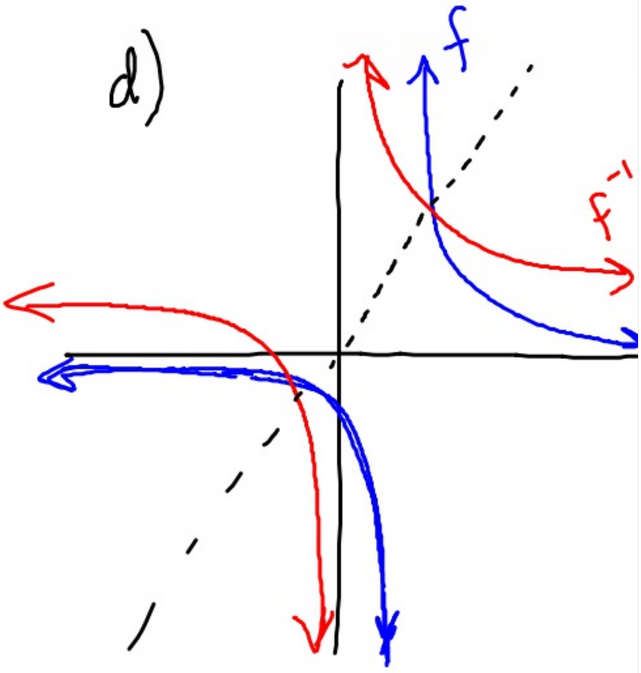
(b) $(y-1)x = \frac{3}{\cancel{y-1}}\cancel{(y-1)}$

$xy - x = 3$

$xy = x + 3$

$y = \frac{x+3}{x}$

d)



Example 6

The air conditioner repair service charges \$75 for coming to your home plus another \$30 per hour.

a) Write a function for the cost, C, of getting your a/c repaired where x is hours. $C(x) =$ $30x + 75$

(input-hours, output-cost)

b) Find the inverse of the cost function in part a.

$$C^{-1}(x) = \frac{x - 75}{30}$$

c) Describe in words what the inverse function does in the context of this problem.

when you input total cost, the output will be hours worked